MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 19: Exponential Generating Functions II

Let us begin by generalizing the combinatorial explanation given in the previous lecture for the product of two exponential generating functions.

Proposition 1. *Fix $k \in \mathbb{N}_{\ge 2}$, and for each $i \in [k]$, let $a_n^{(i)}$ be the number of ways to build certain α_i-structure on an n-set. Let f_n be the number of ways to subdivide $[n]$ into k disjoint subsets, namely, $[n] = T_1 \cup \cdots \cup T_k$, and then build an α_i-structure on T_i for every $i \in [k]$. Let $A_i(x)$ be the exponential generating function of $(a_n^{(i)})_{n\ge 0}$ for every $i \in [k]$, and let $F(x)$ be the exponential generating function $(f_n)_{n\ge 0}$. Then $F(x) = A_1(x)A_2(x)\cdots A_k(x)$.*

Proof. First, we observe that

$$f_n = \sum_{(T_1,\dots,T_k)} a^{(1)}_{|T_1|} \cdots a^{(k)}_{|T_k|} = \sum_{(t_1,\dots,t_k)} \binom{n}{t_1,\dots,t_k} a^{(1)}_{t_1} \cdots a^{(k)}_{t_k},$$

where $(T_1,\dots,T_k)$ runs over all k-tuples of disjoint subsets of $[n]$ with $T_1 \cup \cdots \cup T_k = [n]$ while $(t_1,\dots,t_k)$ runs over all k-tuples of $\mathbb{N}_0^k$ with $t_1 + \cdots + t_k = n$. Thus,

$$F(x) = \sum_{n=0}^{n} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \Bigg(\sum_{(t_1,\dots,t_k)} \binom{n}{t_1,\dots,t_k} a^{(1)}_{t_1} \cdots a^{(k)}_{t_k} \Bigg) \frac{x^n}{n!} = \prod_{i=1}^{k} A_1(x) \cdots A_k(x).$$

□

We are in a position to prove the composition theorem in the context of exponential generating functions.

Theorem 2. *Let a_n be the number of ways to build a certain α-structure on an n-set, and assume that $a_0 = 0$. Let b_n be the number of ways to build a certain β-structure on an n-set, and assume that $b_0 = 1$. Let f_n be the number of ways to partition $[n]$, build an α-structure on each block of the partition, and then build a β-structure on the set of all blocks. Assume that $f_0 = 1$, and let $A(x)$, $B(x)$, and $F(x)$ be the exponential generating functions of $(a_n)_{n\ge 0}$, $(b_n)_{n\ge 0}$, and $(f_n)_{n\ge 0}$, respectively. Then $F(x) = B(A(x))$.*

Proof. Let $\prod_n$ be the set of partitions of the set $[n]$. Observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} f_n &= \sum_{k=1}^{n} b_k \sum_{\{T_1,\dots,T_k\}\in\prod_n} a_{|T_1|}\cdots a_{|T_k|} = \sum_{k=1}^{n} \frac{b_k}{k!} \sum_{(T_1,\dots,T_k)} a_{|T_1|}\cdots a_{|T_k|} \\ &= \sum_{k=1}^{n} \frac{b_k}{k!} \sum_{(t_1,\dots,t_k)} \binom{n}{t_1,\dots,t_k} a_{t_1}\cdots a_{t_k}, \end{aligned}$$

where $(T_1,\dots,T_k)$ runs over all k-tuples of disjoint and nonempty subsets of $[n]$ with $T_1 \cup \dots \cup T_k = [n]$ while $(t_1,\dots,t_k)$ runs over all k-tuples of $\mathbb{N}^k$ with $t_1 + \dots + t_k = n$. As a result,

$$\begin{aligned} F(x) &= 1 + \sum_{n=1}^{\infty} f_n \frac{x^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{b_k}{k!} \sum_{(t_1,\dots,t_k)} \binom{n}{t_1,\dots,t_k} a_{t_1}\cdots a_{t_k} \right) \frac{x^n}{n!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{b_k}{k!} \sum_{n=1}^{\infty} \left(\sum_{(t_1,\dots,t_k)} \binom{n}{t_1,\dots,t_k} a_{t_1}\cdots a_{t_k} \right) \frac{x^n}{n!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{b_k}{k!} \sum_{n=1}^{\infty} \left(\sum_{(t_1,\dots,t_k)} \frac{a_{t_1}}{t_1!} \cdots \frac{a_{t_k}}{t_k!} \right) x^n \\ &= 1 + \sum_{k=1}^{\infty} \frac{b_k}{k!} A(x)^k = B(A(x)). \end{aligned}$$

□

We conclude this lecture with the following example.

Example 3. In how many ways we can seat a group of n people at unspecified number of circular tables. With notation as in the previous theorem, we see that $a_k = (k-1)!$ for every $k \in \mathbb{N}$ and $a_0 = 0$. Therefore

$$A(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty} \int x^{k-1} dx = \int \left(\sum_{k=1}^{\infty} x^{k-1} \right) dx = \int \frac{dx}{1-x} = \ln\left(\frac{1}{1-x}\right).$$

On the other hand, we can take $b_n = 1$ for every $n \in \mathbb{N}_0$ as we are doing nothing with the collection of tables. Hence

$$B(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Since

$$B(A(x)) = e^{\ln\left(\frac{1}{1-x}\right)} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n!\frac{x^n}{n!},$$

by virtue of the composition theorem the desired number is $f_n = n!$. This is hardly a surprise, as we can think of each arrangement of the n people at circular tables as a disjoint cycle decomposition of a permutation of the set consisting of the n people.

Practice Exercises

Exercise 1. [1, Exercise 8.47] *We divide a group of of people into subgroups A, B, and C, and ask each subgroup to form a line. In addition, we want A to have odd size and B to have even size. In how many ways can we do this?*

Exercise 2. [1, Exercise 8.24] *Let g_n be the number of ways to choose a partition of $[n]$ into blocks of size at most two, and then to linearly order the set of blocks. Find an explicit formula for the exponential generating function of $(g_n)_{n\geq 0}$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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